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Tabulation of Constants for Full Grade I_{MN} Approximants

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Abstract. For f: $[0, \infty) \longrightarrow R$ the I_{MN} approximant of $f(t)$ is

$$I_{MN}(f, t) = \int_0^\infty f(xt) \sum_{i=1}^{N} K_i e^{-\alpha_i x}\, dx,$$

where α_i, K_i are defined constants. Under appropriate conditions on f, I_{MN} approximants of full grade are capable of giving good approximation both for small and large t. These and other properties of full grade I_{MN} approximants make them particularly useful in a wide range of practical applications. The constants α_i, K_i of full grade I_{MN} approximants are generated by partial fraction decompositions of certain Padé approximants to e^{-z}. The purpose of this paper is firstly to tabulate the constants α_i, K_i for all full grade I_{MN} approximants for $1 \leqslant N \leqslant 10$; secondly, to give accurate estimates of their precision; and thirdly, to describe the methods of tabulation and estimation in sufficient detail to allow the results of this paper to be extended readily.

1. Introduction and Review. Let F denote the linear space of functions f: $[0, \infty) \longrightarrow R$ such that f is continuous on $[0,\infty)$ and such that, for some real σ, $f(\lambda) = O(e^{\sigma\lambda})$, $\lambda \longrightarrow \infty$. Let $I_{MN}(f, t)$ denote the improper integral

$$I_{MN}(f, t) = \int_0^\infty f(xt) \sum_{i=1}^{N} K_i e^{-\alpha_i x}\, dx, \qquad t \in T, \tag{1.1}$$

where T is the set of all $t \in [0,\infty)$ such that the improper integral converges to a finite limit. $I_{MN}(f, t)$ is said to be the I_{MN} approximant of f evaluated at t.

Let $I_{MN}f$ denote the function $t \longmapsto I_{MN}(f, t)$ which maps T into R and let I_{MN} denote the operator $f \longmapsto I_{MN}f$ which maps F into some set of functions. Ideally, it is required that I_{MN} be equal to the identity operator I: $F \longrightarrow F$ in which case $I_{MN}(f, t) = f(t)$ for all $f \in F$ and all $t \geqslant 0$. Since this is not possible, it is instead required to choose the constants α_i, K_i so that I_{MN} is, in some sense, the best approximant to I. Several criteria of best approximation have been defined [14], [19] and to each criterion there corresponds one set of constants α_i, K_i. One particular criterion defines the class of full grade I_{MN} approximants [18], [19] which have remarkable properties in certain applications.

Let

$$\hat{\alpha}_{MN} = \min_i \{\operatorname{Re}(\alpha_i)\}. \tag{1.2}$$

For the purpose of this paper it is convenient to define [19], [22] the class

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of full grade approximants as all those I_{MN} such that the constants α_i, K_i satisfy the relations

$$\sum_{i=1}^{N} \frac{K_i}{z+\alpha_i} = \frac{A_{MN}(z)}{B_{MN}(z)}, \tag{1.3}$$

$$B_{MN}(z) = \sum_{k=0}^{N} \frac{(M+N-k)!}{(M+N)!} \frac{N!}{(N-k)!} \frac{z^k}{k!}, \qquad A_{MN}(z) = B_{NM}(-z) \tag{1.4}$$

and $\hat{\alpha}_{MN} > 0$.

The existence of numbers α_i, K_i that satisfy (1.3) has been established [19]. Notice that the rational function $A_{MN}(z)/B_{MN}(z)$ is the (M/N) Padé approximant of e^{-z}. Clearly, given that the α_i, K_i satisfy (1.3), the I_{MN} approximant has full grade if, and only if, $\hat{\alpha}_{MN} > 0$. This condition is known to hold for all (M, N) such that $N - 4 \leqslant M \leqslant N - 1$ [8]. For other values of (M, N) it is easy to test whether $\hat{\alpha}_{MN} > 0$. A table of such results is given in [19] for all (M, N) such that $0 \leqslant M < N \leqslant 20$; moreover, for all (M, N) such that $0 \leqslant M < N \leqslant 10$, $\hat{\alpha}_{MN} > 0$ if, and only if, the corresponding entry in Tables 1–3 of this paper lies above the "staircase".

An extensive study of the properties of I_{MN} approximants in general, and of full grade approximants in particular, has been made [18], [19]. The main results relating to full grade approximants are summarized in the following theorem; it should be noted that many of these results hold only if $\hat{\alpha}_{MN} > 0$.

Let $T_n = \{t\colon \hat{\alpha}_{MN} > \sigma_n t,\ t \geqslant 0\}$ where $\sigma_n = \inf\{\sigma\colon f^{(k)}(\lambda) = O(e^{\sigma\lambda}),\ \lambda \longrightarrow \infty,\ k = 0, 1, \ldots, n\}$.

THEOREM 1.1 [18], [19]. (a) *Let $f \in F$. Then $I_{MN}(f, t)$ exists for all $t \in T_0$. In particular, if f is such that $f(\lambda) = O(e^{\sigma\lambda})$, $\lambda \longrightarrow \infty$, for all $\sigma > 0$, then $I_{MN}(f, t)$ exists for all $t \geqslant 0$.*

(b) *Let $f \in F$. Then $I_{MN}f$ is analytic in $T_0 - \{0\}$.*

(c) *Let $f \in F$ be bounded on $[0, \infty)$. Then $I_{MN}f$ is bounded and continuous on $[0, \infty)$. This means that the operator I_{MN} maps the space of bounded and continuous functions into itself.*

(d) *Let f be a polynomial of degree not greater than $M + N$. Then*

$$I_{MN}(f, t) = f(t) \quad \text{for all } t \geqslant 0.$$

(e) *Let $f \in F$ and $\sigma_0 < 0$. Then*

$$\begin{aligned} I_{MN}(f, t) &= O(t^{-(N-M)}), \qquad t \longrightarrow \infty, \\ f(t) - I_{MN}(f, t) &= O(t^{-(N-M)}), \qquad t \longrightarrow \infty. \end{aligned} \tag{1.5}$$

(f) *Let $f \in F$. Assume f is bounded on $[0, \infty)$ and $\lim_{t\to\infty} f(t)$ exists. Then*

$$\lim_{t\to\infty} I_{MN}(f, t) = \lim_{t\to\infty} f(t).$$

(g) *Let $f \in F$. Then*

$$\lim_{t\to 0+} I_{MN}(f, t) = I_{MN}(f, 0) = f(0).$$

(h) *Let* $f \in F$. *Assume that there is a* $\tau > 0$ *such that* $f^{(M+N)}$ *is continuous on* $[0, \tau]$, $f^{(M+N+1)}$ *exists everywhere in* $(0, \tau)$ *and is bounded on* $(0, \tau)$. *Then*

$$f(t) - I_{MN}(f, t) = O(t^{M+N+1}), \qquad t \longrightarrow 0+. \tag{1.6}$$

(i) *Let* $f^{(k)} \in F$ *for* $k = 0, 1, \ldots, M + N$, *and let* $f^{(M+N+1)}$ *be continuous on* $[0,\infty)$. *Let* $t \in T_{M+N}$. *Then*

$$I_{MN}(f, t) = \sum_{k=0}^{M+N} \frac{t^k}{k!} f^{(k)}(0) + R_{MN}(t),$$

$$R_{MN}(t) = \int_0^\infty f^{(M+N+1)}(xt) \sum_{i=1}^{N} \frac{K_i t^{M+N+1}}{\alpha_i^{M+N+1}} e^{-\alpha_i x}\, dx.$$

(j) *Let* $f^{(k)} \in F$, $k = 0, 1, \ldots, M + N + 1$. *Then*

$$R_{MN}(t) = O(t^{M+N+1}), \qquad t \longrightarrow 0+.$$

These properties are important in the approximation of a function on the entire half-line $[0, \infty)$. In particular, (1.5) ensures good accuracy for large t while (1.6) ensures good accuracy for small t. By changing M, a tradeoff is effected between accuracy at large t and accuracy at small t.

Let $L(f, s)$ denote the Laplace transform of f evaluated at s, that is

$$L(f, s) = \int_0^\infty f(\lambda) e^{-s\lambda}\, d\lambda.$$

From (1.1) it follows readily that

$$I_{MN}(f, t) = t^{-1} \sum_{i=1}^{N} K_i L(f, \alpha_i/t), \qquad t \in T - \{0\}. \tag{1.7}$$

Equation (1.7) provides a useful quadrature formula for the numerical inversion of Laplace transforms.

A brief review of applications of full grade I_{MN} approximants communicated up to the year 1973 is included in [19]. It is sufficient to mention here that the approximants give rise to efficient recursive techniques for the solution of initial value problems in first-order linear constant ordinary differential equations [15], [16], [19] and the numerical inversion of rational Laplace transforms [17]. More recently, the recursions have been extended to cover first-order linear differential-algebraic systems [20] and high-order linear differential and differential-algebraic systems [23]. With the recursive methods, small values of N may be used; in practice, the values $N = 4$ and $N = 6$ are found to give sufficient accuracy. This is significant because, with single-precision arithmetic (10 figures), values of N not greater than 10 generally must be used in order to avoid excessive roundoff errors due to cancellation. For an important class of problems, including, for example, diffusion problems, the function f is sufficiently smooth on $[0, \infty)$ for the formula (1.7) to be applied in a global fashion tc invert the Laplace transform $L(f, s)$ at a number of values of t [19]. It is found in this case that sufficient accuracy is attained with $N = 10$. Further practical applicatioı of both recursive and global techniques have been reported recently [1], [2], [13].

The inversion formula (1.7) has also been obtained as an approximation to the Bromwich integral

$$f(t) = \frac{1}{2\pi i t} \int_C L(f, \alpha/t) e^{\alpha}\, d\alpha,$$

where C is any line $\mathrm{Re}(\alpha) = \sigma_0 t + \epsilon$, $\epsilon > 0$. Two approaches have been used:

(i) *Gaussian quadrature* [9], [10], [12], [4], [5], [6]. The formula (1.7) is a Gaussian quadrature of the Bromwich integral for all (M, N) such that $M = N - 1$. The α_i are obtained as the reciprocal of the zeros of orthogonal Bessel polynomials, while the K_i can be expressed in terms of the Christoffel numbers. It has been shown [19] that the inversion formula so obtained corresponds to the full grade $I_{N-1,N}$ approximant.

(ii) *Approximation of the term e^{α} in the Bromwich integral by its Padé approximant expressed in partial fractions* [11]. The connection with I_{MN} approximants is easy to show.

These alternative approaches have yet to reveal properties of I_{MN} approximants such as those listed in Theorem 1.1.

Previous tabulations of the constants α_i, K_i include the following: Salzer gives the α_i and K_i for $M = N - 1$, $N = 1(1)8$ to between 4 and 8 significant figures [9] and for $M = N - 1$, $N = 1(1)16$ to between 12 and 15 significant figures [10]. Stroud and Secrest [12] give the α_i and K_i/α_i for $M = N - 1$, $N = 2(1)24$ to 30 significant figures (computations performed using 39-decimal arithmetic). Krylov and Skoblya [4] give the α_i and K_i/α_i for $M = N - 1$, $N = 1(1)15$ to 20 significant figures (computations performed in triple-precision arithmetic on a MINSK2 computer). Rodrigues [7] gives the α_i and K_i to between 6 and 10 significant figures for $(M, N) = (5, 10)$ and $(3, 10)$. Notice that for $(M, N) = (3, 10)$ the I_{MN} approximant does not have full grade since $\hat{\alpha}_{MN} < 0$.

The purpose of this paper is threefold.

(i) To tabulate the constants α_i, K_i for all full grade I_{MN} approximants such that $N = 1(1)10$.

(ii) To give accurate estimates of their precision. These estimates are based on tests applied separately to each constant, in contrast with methods used by other authors involving the evaluation of a function, whose value is known, of all the constants in each set.

(iii) To describe the methods of tabulation and estimation in sufficient detail to allow the results of this paper to be extended readily.

A more detailed account of the methods of computation and estimation, together with the results of a comparison of several methods of computation, are given in [21].

2. Method of Computation. The constants tabulated were computed on a DEC PDP-10 computer using 15-decimal floating arithmetic.

Let

$$A_{MN}(z) = \sum_{k=0}^{M} a_k z^k, \quad B_{MN}(z) = \sum_{k=0}^{N} b_k z^k.$$

The coefficients a_k, b_k are determined readily from the recurrence relations

$$\frac{a_{k+1}}{a_k} = \frac{-(M-k)}{(M+N-k)(k+1)}, \qquad k = 0, 1, \ldots, M-1,$$

$$\frac{b_{k+1}}{b_k} = \frac{N-k}{(M+N-k)(k+1)}, \qquad k = 0, 1, \ldots, N-1,$$

with $a_0 = b_0 = 1$.

Clearly, the $-\alpha_i$ are the zeros of the polynomial $B_{MN}(z)$. These are determined using successive quadratic factorization performed by Bairstow iteration. The round-off error generated by this process may be reduced by a transformation of the form $z = cs$, where c is a positive constant. Numerical tests show that the best accuracy is obtained when c is chosen so as to minimize the coefficient spread of the transformed polynomial, that is $c = b_N^{-1/N}$.

The K_i are given by

$$K_i = \frac{A_{MN}(-\alpha_i)}{b_N \Pi_{j=1;j\neq i}^{N}(\alpha_j - \alpha_i)}. \tag{2.1}$$

Alternative expressions for K_i follow readily from (2.1) and the relations

$$b_N \prod_{j=1;j\neq i}^{N} (\alpha_j - \alpha_i) = B_{MN}^{(1)}(-\alpha_i) \quad \text{and} \quad A_{MN}(-\alpha_i) = \frac{(-1)^{N+1}|a_M|b_N\alpha_i^{M+N}}{B_{MN}^{(1)}(-\alpha_i)},$$

where the superscript $^{(1)}$ denotes the first derivative.

Of four formulae thus obtained, (2.1) is found generally to give rise to the least error in K_i due both to errors in the α_i and to roundoff in the evaluation of the formula (see [21]).

3. **Assessment of Accuracy**. The number of significant figures of agreement between two floating point numbers a, b is defined as follows:

Let

$$10^{p-1} \leqslant |a| < 10^p, \qquad 10^{q-1} \leqslant |b| < 10^q, \qquad |a| \leqslant |b|.$$

Then a and b are said to agree to k significant figures if, and only if, $.5 \times 10^{q-k-1} < |b-a| \leqslant .5 \times 10^{q-k}$.

3.1. *Accuracy of the* α_i. Let $\alpha_i + \delta\alpha_i$, $i = 1, 2, \ldots, N$, denote the computed values of the α_i. Then there exists a polynomial $\Sigma_{k=0}^N (b_k + \epsilon_k)z^k$ such that $\Sigma_{k=0}^N (-1)^k (b_k + \epsilon_k)(\alpha_i + \delta\alpha_i)^k = 0$, $i = 1, 2, \ldots, N$. The coefficients $b_k + \epsilon_k$ are readily computed from the $\alpha_i + \delta\alpha_i$; $\delta\alpha_i$ is given by

$$\delta\alpha_i = \frac{\Sigma_{k=0}^N \epsilon_k(-\alpha_i)^k}{\Sigma_{k=1}^N k b_k(-\alpha_i)^{k-1}} + E, \qquad i = 1, 2, \ldots, N, \tag{3.1}$$

TABLE 1

Accuracy of α_i: $\min_i\{n_i\}$ where n_i is an estimate of the number of significant figures of agreement between computed and exact values of $|\alpha_i|$.

N \ M	0	1	2	3	4	5	6	7	8	9
1	15									
2	15	15								
3	15	15	15							
4	14	15	15	15						
5		15	15	14	14					
6		15	14	14	14	14				
7			14	14	12	14	13			
8				14	13	13	13	13		
9				13	13	13	12	13	13	
10					12	12	12	13	12	12

where E denotes an expression in the second and higher powers of ϵ_k, $\delta\alpha_i$ which may be neglected when the ϵ_k are sufficiently small. Using computed values of the α_i, an estimate of $\delta\alpha_i$, and therefore an estimate of number n_i of significant figures of agreement between the exact and computed values of $|\alpha_i|$, is obtained by means of (3.1). The least value of n_i, for each (M, N) such that $0 \leqslant M < N \leqslant 10$, is shown in Table 1.

3.2. *Accuracy of the* K_i. The error in the computed value of K_i arises from two sources:

(i) *Inherited error* δK_i *caused by errors* $\delta\alpha_j$ *in the computed values of the* α_j. From (2.1), the inherited error δK_i is given by

$$\delta K_i = K_i\left(\frac{\sum_{k=1}^{M}(-1)^k k a_k \alpha_i^k}{\sum_{k=0}^{M}(-1)^k a_k \alpha_i^k}\,\frac{\delta\alpha_i}{\alpha_i} - \sum_{j=1;j\neq i}^{N}\frac{\delta\alpha_j - \delta\alpha_i}{\alpha_j - \alpha_i}\right) + E, \qquad i = 1, 2, \ldots, N,$$

where E denotes an expression in the second and higher powers of the $\delta\alpha_j$ which is neglected when the $\delta\alpha_j$ are sufficiently small. An estimate of δK_i is obtained using computed values of the α_i and $\delta\alpha_i$.

(ii) *Roundoff error* R_i *generated in the evaluation of* $K_i + \delta K_i$ *from the computed values of the* α_j. Using the method of forward error analysis (see, for example, [3, pp. 15–25]), and regarding each complex operation as a pair of real operations, an upper bound $\hat{R}_i$ on $|R_i|$ may be estimated.

An estimated upper bound on the total error in K_i is then given by $\hat{R}_i + |\delta K_i|$. Using this quantity, an estimate may be made of a lower bound n_i on the number of significant figures of agreement between the exact and computed values of $|K_i|$. The least value of n_i for each (M, N) such that $0 \leqslant M < N \leqslant 10$ is shown in Table 2.

TABLE 2

Accuracy of K_i: $\min_i\{n_i\}$, where n_i is an estimated lower bound on the number of significant figures of agreement between computed and exact values of $|K_i|$.

N \ M	0	1	2	3	4	5	6	7	8	9
1	15									
2	14	14								
3	13	13	13							
4	13	13	13	13						
5		13	13	12	13					
6		12	13	12	12	12				
7			12	13	11	12	12			
8				12	12	12	12	12		
9				12	12	12	12	11	11	
10					11	12	11	11	11	11

TABLE 3

Error in c_0: $\max_{\Delta\in\Omega}(\Delta - n_0)$, where n_0 is the number of significant figures of agreement between c_0 and $c_0(\Delta)$ and $\Omega = \{7, 8, 10, 12, 14, 15\}$.

N \ M	0	1	2	3	4	5	6	7	8	9
1	0									
2	1	1								
3	0	1	1							
4	1	1	1	2						
5		2	2	3	3					
6		2	1	2	2	3				
7			2	3	3	4	4			
8				2	3	3	3	4		
9				3	4	4	4	5	5	
10					3	4	4	4	4	5

3.3. *Estimate of Error in Computing* $I_{MN}(f, t)$. It is known [19] that an estimate of the error generated by the quadrature inversion formula (1.7) is given by computing the number c_0, where $c_0 = \Sigma_{i=1}^N K_i/\alpha_i$ and has the value 1 for all (M, N) such that $0 \leqslant M < N$. If $c_0(\Delta)$ denotes the value of c_0 computed with Δ-decimal floating arithmetic, then an estimate of the error is given by $1 - c_0(\Delta)$. Let n_0 denote the number of significant figures of agreement between 1 and $c_0(\Delta)$.

TABLE 4

Tables of constants α_i, K_i for all full-grade I_{MN} approximants, $1 \leqslant N \leqslant 10$

APPROXIMANT	I	RE(ALPHA(I))	IM(ALPHA(I))	RE(K(I))	IM(K(I))
$I_{0,1}$	1	0.1000000000000000D+01	0.0000000000000000D+00	0.1000000000000000D+01	0.0000000000000000D+00
$I_{0,2}$	1	0.1000000000000000D+01	0.9999999999999999D+00	0.0000000000000000D+00	0.1000000000000000D+01
$I_{1,2}$	1	0.2000000000000000D+01	0.1414213562373096D+01	-0.1000000000000000D+01	0.3535533905932737D+01
$I_{0,3}$	1	0.7019641810083393D+00	0.1807339494452022D+01	-0.7378432588978603D+00	0.3650178408010285D+00
	3	0.1596071637983322D+01	0.0000000000000000D+00	0.1475686517795721D+01	0.0000000000000000D+00
$I_{1,3}$	1	0.1687091590520765D+01	0.2508731754924880D+01	-0.2770399509339410D+01	-0.1591864442851175D+00
	3	0.2625816818958471D+01	0.0000000000000000D+00	0.5540799018678817D+01	0.0000000000000000D+00
$I_{2,3}$	1	0.2681082873627753D+01	0.3050430199247411D+01	-0.7648749087422927D+01	-0.4171640244747441D+01
	3	0.3637834252744495D+01	0.0000000000000000D+00	0.1829749817484585D+02	0.0000000000000000D+00
$I_{0,4}$	1	0.1729444231067707D+01	0.8889743761218606D+00	0.5414133484291550D+00	0.1588859182223285D+01
	3	0.2705557689322938D+00	0.2504775904362437D+01	-0.5414133484291552D+00	-0.2485625208661186D+00
$I_{1,4}$	1	0.2764346415715101D+01	0.1162323629283277D+01	0.7432425057989014D+00	0.6054583528372886D+01
	3	0.1235653584284900D+01	0.3437652493671051D+01	-0.7432425057989023D+00	-0.1716635413478418D+01
$I_{2,4}$	1	0.3779019967010191D+01	0.1380176524272846D+01	-0.1128479372209080D+01	0.1981654350025087D+02
	3	0.2220980032989810D+01	0.4160391445506932D+01	0.1128479372209078D+01	-0.5554415818937960D+01
$I_{3,4}$	1	0.4787193103128468D+01	0.1567476416895205D+01	-0.1330153999597150D+02	0.6007173273704761D+02
	3	0.3212806896871536D+01	0.4773087433276642D+01	0.1130153999597149D+02	-0.1247167585025024D+02
$I_{1,5}$	1	0.2678150385698260D+01	0.2181220591827919D+01	-0.5052900858664491D+01	0.2546761572279270D+01
	3	0.7032932336840762D+00	0.4260145197669165D+01	0.6037536657992389D+00	-0.1000076276081612D+01
	5	0.3237112761235333D+01	0.0000000000000000D+00	0.8898294385730502D+01	0.0000000000000000D+00
$I_{2,5}$	1	0.3690075242282725D+01	0.2571878530841120D+01	-0.1784911957917488D+02	0.2475027299891130D+01
	3	0.1678685104535973D+01	0.5138525539012343D+01	0.3307264349921304D+01	-0.9134627052229177D+00
	5	0.4262479306362613D+01	0.0000000000000000D+00	0.2908371045850713D+02	0.0000000000000000D+00
$I_{3,5}$	1	0.4696707076835910D+01	0.2908975454213617D+01	-0.5444764340490001D+02	-0.8595283832952380D+01
	3	0.2664731518065843D+01	0.5884022927615474D+01	0.9226960955549153D+01	0.3824136376144439D+01
	5	0.5277122810196498D+01	0.0000000000000000D+00	0.9044136489870171D+02	0.0000000000000000D+00
$I_{4,5}$	1	0.5700953298671781D+01	0.3210265600308557D+01	-0.1499984465975463D+03	-0.6804227952202188D+02
	3	0.3555694326463576D+01	0.6543736899360081D+01	0.1582680186458594D+02	0.2412564578224447D+02
	5	0.6233670475172929D+01	0.0000000000000000D+00	0.2733432894659208D+03	0.0000000000000000D+00
$I_{1,6}$	1	0.2458301063997515D+01	0.3102173276343168D+01	-0.3835650169866717D+01	-0.2693892062542721D+01
	3	0.3424888194918926D+01	0.1047550917430439D+01	0.3156897931331429D+01	0.9968268228850436D+01
	5	0.1168107410835596D+00	0.5006586267818796D+01	0.6787522385352839D+00	-0.1244807166685768D+00

TABLE 4 (*continued*)

APPROXIMANT	I	RE(ALPHA(I))	IM(ALPHA(I))	RE(K(I))	IM(K(I))
$I_{2,6}$	1	0.3464309367148108D+01	0.3639365824336670D+01	-0.6953356360348368D+01	-0.1282520511113620D+02
	3	0.4454039259447837D+01	0.1217794942586588D+01	0.5461118730282307D+01	0.3347016373960818D+02
	5	0.1081651373404057D+01	0.6023443092124340D+01	0.1492237630066050D+01	0.1289189166077830D+01
$I_{3,6}$	1	0.5471177123624819D+01	0.1366028955687703D+01	0.3303965510117951D+01	0.1049213413552316D+03
	3	0.4467389858351445D+01	0.4103985176828966D+01	-0.3760525748774691D+01	-0.4381430805831008D+02
	5	0.2061433018023737D+01	0.6887098657124514D+01	0.4565602386567195D+00	0.5620029078309140D+01
$I_{4,6}$	1	0.5469259464575019D+01	0.4519930803296233D+01	0.3715697614716392D+02	-0.1246338304915599D+03
	3	0.6482534491904163D+01	0.1499310620485981D+01	-0.2784121003925132D+02	0.3150653096858251D+03
	5	0.3048206043520819D+01	0.7651791019757355D+01	-0.9315766107912749D+01	0.1310787094104739D+02
$I_{5,6}$	1	0.7490637528809645D+01	0.1621502388778377D+01	-0.1855448788575486D+03	0.9177923648638177D+03
	3	0.6470514936701526D+01	0.4900121147421393D+01	0.2260604118936238D+03	-0.3051033590450531D+03
	5	0.4038847534488837D+01	0.8345600414872205D+01	-0.4351553303607540D+02	0.1401530621564048D+02
$I_{2,7}$	1	0.4469576756913928D+01	0.2324764925255292D+01	-0.3043484247864497D+02	0.1553555886462892D+02
	3	0.3146407784919013D+01	0.4616377954762942D+01	0.6118018860356861D+01	-0.9002065631119462D+01
	5	0.4450793190650611D+00	0.6837867762506168D+01	-0.3351229762906064D-01	0.1088703340526335D+01
	7	0.4877872278204118D+01	0.0000000000000000D+00	0.4371067183183429D+02	0.0000000000000000D+00
$I_{3,7}$	1	0.5486452128620962D+01	0.2598110585275065D+01	-0.1021541006373553D+03	0.2383302402494403D+02
	3	0.4144625920277145D+01	0.5193543854077737D+01	0.2900302266855812D+02	-0.1427025443955509D+02
	5	0.1418330172605530D+01	0.7820946712990146D+01	-0.2435130293669272D+01	0.1860651374615762D+01
	7	0.5901183556992729D+01	0.0000000000000000D+00	0.1511724165249328D+03	0.0000000000000000D+00
$I_{4,7}$	1	0.6497699015561534D+01	0.2844423987709969D+01	-0.3142266245051513D+03	0.2022730851932123D+01
	3	0.5143170117030608D+01	0.5710836470301287D+01	0.9457246119433242D+02	0.1277631566107517D+01
	5	0.2400683827893845D+01	0.8669076365347985D+01	-0.8553194553354737D+01	-0.1204859030015230D+01
	7	0.6916894079028034D+01	0.0000000000000000D+00	0.4564147157283473D+03	0.0000000000000000D+00
$I_{5,7}$	1	0.7505760833151439D+01	0.3070613819912885D+01	-0.9076420055478994D+03	-0.1848062375361909D+03
	3	0.6142001422691473D+01	0.6184072511919049D+01	0.2467363886672691D+03	0.1190077626204650D+03
	5	0.3388117940767962D+01	0.9449711101874953D+01	-0.1576471516086237D+02	-0.1853821743730273D+02
	7	0.7928239606778260D+01	0.0000000000000000D+00	0.1353340664082986D+04	0.0000000000000000D+00
$I_{6,7}$	1	0.8511834825102658D+01	0.3281013624324833D+01	-0.2490669424454205D+04	-0.1040334617478794D+04
	3	0.7141055219187743D+01	0.6623045922639339D+01	0.5152296237316458D+03	0.5954681369906201D+03
	5	0.4378693561506783D+01	0.1016969328379504D+02	-0.2052785571708928D+01	-0.6844192002362125D+02
	7	0.8936832788405645D+01	0.0000000000000000D+00	0.3961985172588534D+04	0.0000000000000000D+00
$I_{3,8}$	1	0.5380220301060496D+01	0.3731789705997154D+01	-0.4918993997107876D+02	-0.8047452217632426D+02
	3	0.6120827047801732D+01	0.1245879601465620D+01	0.3287773645622125D+02	0.1793541699761208D+03
	5	0.3754020493139361D+01	0.6202257571660542D+01	0.1786962903277549D+02	0.1340115391913565D+02
	7	0.7449321579984274D+00	0.8666742100906204D+01	-0.1557425517918028D+01	-0.4324709914596024D+00
$I_{4,8}$	1	0.6389988250001529D+01	0.4077455339604063D+01	-0.6498613253456045D+02	-0.2728648924922098D+03
	3	0.7138702005571693D+01	0.1358287007691404D+01	0.4349423858270048D+02	0.5479005509447996D+03
	5	0.4748575834563451D+01	0.6811186613752122D+01	0.2533164469859634D+02	0.6031178535281835D+02
	7	0.1722730309863332D+01	0.9614935083787837D+01	-0.1839750746736640D+01	-0.4113604956648559D+01

TABLE 4 (*continued*)

APPROXIMANT	I	RE(ALPHA(I))	IM(ALPHA(I))	RE(K(I))	IM(K(I))
$I_{5,8}$	1	0.7396983588257334D+01	0.4395261731624626D+01	0.4853922455717320D+02	-0.8208651118932657D+03
	3	0.8151713382166029D+01	0.1461831479154053D+01	-0.2599458700064747D+02	0.1627630214685013D+04
	5	0.5744470296037129D+01	0.7368613908009308D+01	-0.2726577045334811D+02	0.1837719901597092D+03
	7	0.2706832733539526D+01	0.1047568288041923D+02	0.4721132896821745D+01	-0.1167022539644645D+02
$I_{6,8}$	1	0.8402252985149454D+01	0.4691162756053741D+01	0.7826848084845861D+03	-0.2253940546158688D+04
	3	0.9161628867584762D+01	0.1558377394062391D+01	-0.5017155587577998D+03	0.4734997309800120D+04
	5	0.6741261817878945D+01	0.7885949689485654D+01	-0.3128179760333806D+03	0.4285946133749973D+03
	7	0.3694856329386854D+01	0.1126969930009705D+02	0.3184872630659417D+02	-0.1495281866236859D+02
$I_{7,8}$	1	0.9406371213690576D+01	0.4969217287623496D+01	0.3848544415437202D+04	-0.5690315433805139D+04
	3	0.1016944600665763D+02	0.1649201796822954D+01	-0.2612620846867762D+04	0.1354906014743459D+05
	5	0.7738688146830600D+01	0.8370879306237684D+01	-0.1334864737460233D+04	0.7024310104983031D+03
	7	0.4685494632821201D+01	0.1201057859981383D+02	0.9494116889078772D+02	0.2547666248177078D+02
$I_{3,9}$	1	0.6196445435953924D+01	0.2401875514215652D+01	-0.1726518359055972D+03	0.8867674741411167D+02
	3	0.5185206541695638D+01	0.4788776840818324D+01	0.4476843215671663D+02	-0.6426784524891331D+02
	5	0.3311558690983490D+01	0.7146312283966383D+01	-0.1399254115898873D+01	0.1463548028463564D+02
	7	0.4755565177352481D-01	0.9472329433888902D+01	-0.3528597170160872D+00	-0.7533065574237304D+00
	9	0.6518467359186859D+01	0.0000000000000000D+00	0.2592710351635912D+03	0.0000000000000000D+00
$I_{4,9}$	1	0.7214820536441866D+01	0.2612611735839767D+01	-0.5597834098688290D+03	0.1621691058000984D+03
	3	0.6192489096837279D+01	0.5223853880816573D+01	0.1953392240980158D+03	-0.1234345334672306D+03
	5	0.4301663288838062D+01	0.7839424199732851D+01	-0.2824082034071143D+02	0.3125646242406460D+02
	7	0.1020722304002136D+01	0.1050267454518626D+02	0.1146637700984899D+01	-0.1988881832731344D+01
	9	0.7540609547761344D+01	0.0000000000000000D+00	0.7830767368210793D+03	0.0000000000000000D+00
$I_{5,9}$	1	0.8228258164127171D+01	0.2807014076996436D+01	-0.1712103829742525D+04	0.1760289630313636D+03
	3	0.7197645205110057D+01	0.5624229214846597D+01	0.6607864579136367D+03	-0.1289493154428165D+03
	5	0.5294239542378278D+01	0.8474226767977425D+01	-0.1164702044407510D+03	0.2911324239728008D+02
	7	0.2001415359570747D+01	0.1143815065195379D+02	0.6272633379207676D+01	-0.1069433285966880D+01
	9	0.8556883457627507D+01	0.0000000000000000D+00	0.2323029885780863D+04	0.0000000000000000D+00
$I_{6,9}$	1	0.9238538967113415D+01	0.2988485840858383D+01	-0.5011973250512248D+04	-0.3341293155159491D+03
	3	0.8201497231129880D+01	0.5997279346337678D+01	0.1920514163496253D+04	0.3012612397959846D+03
	5	0.6288460760809060D+01	0.9063607918550854D+01	-0.3246750839807306D+03	-0.1032335869359864D+03
	7	0.2980813248439626D+01	0.1230122222956795D+02	0.1410260965428005D+02	0.1064851885565978D+02
	9	0.9569379585016062D+01	0.0000000000000000D+00	0.6804063122684893D+04	0.0000000000000000D+00
$I_{7,9}$	1	0.1024667073471309D+02	0.3159363212330701D+01	-0.1415840725963421D+05	-0.3267646525657058D+04
	3	0.9204489309799268D+01	0.6348639013514734D+01	0.4931373075432346D+04	0.2600295640724296D+04
	5	0.7283832885996640D+01	0.9616252725527461D+01	-0.6507470997240015D+03	-0.7104521683802124D+03
	7	0.3975361717884006D+01	0.1310663233194499D+02	0.7792864477064843D+01	0.4849057552744816D+02
	9	0.1057929070321401D+02	0.0000000000000000D+00	0.1973997683889760D+05	0.0000000000000000D+00
$I_{8,9}$	1	0.1125326985727062D+02	0.3321340531523970D+01	-0.3877339741511828D+05	-0.1546808402762370D+05
	3	0.1020688322084996D+02	0.6680140723789168D+01	0.1108964544384748D+05	0.1139639863068603D+05
	5	0.8280042200550302D+01	0.1013835966110886D+02	-0.6631214437594124D+03	-0.2634268181336182D+04
	7	0.4966129260866780D+01	0.1386468597891407D+02	-0.7288777017372871D+02	0.1154258460856603D+03
	9	0.1158735092128469D+02	0.0000000000000000D+00	0.5684852237040787D+05	0.0000000000000000D+00

TABLE 4 (*continued*)

APPROXIMANT	I	RE(ALPHA(I))	IM(ALPHA(I))	RE(K(I))	IM(K(I))
$I_{4,10}$	1	0.7183879675330663D+01	0.3785600991918826D+01	-0.3084074436374916D+03	-0.4694490231355162D+03
	3	0.7777651129281075D+01	0.1262856772620325D+01	0.1851586320219865D+03	0.9476537169849107D+03
	5	0.5926075955646947D+01	0.6300482964122305D+01	0.1479804512205301D+03	0.1002452671105175D+03
	7	0.3813262608129530D+01	0.8808412206350011D+01	-0.2586368146511457D+02	-0.5215010674003745D+01
	9	0.2991306316117935D+00	0.1134191421619468D+02	0.1132041860088995D+01	-0.1822946892791056D+00
$I_{5,10}$	1	0.8196776588963626D+01	0.4061604337786885D+01	-0.5279679746768242D+03	-0.1565549528231606D+04
	3	0.8795804648457035D+01	0.1353404393998887D+01	0.3117445697928050D+03	0.2846035460096570D+04
	5	0.6928803076110155D+01	0.6776484143181877D+01	0.2626748167990334D+03	0.4457572760435456D+03
	7	0.4802247200401481D+01	0.9515150101448705D+01	-0.4867360794506737D+02	-0.5653979539270709D+02
	9	0.1276368486067706D+01	0.1234767915724995D+02	0.2222196030051750D+01	0.2239117823041691D+01
$I_{6,10}$	1	0.9206660348596783D+01	0.4319479994744564D+01	-0.4044078283949647D+03	-0.4789601123626547D+04
	3	0.9809818483222335D+01	0.1438089736235301D+01	0.2518818238440438D+03	0.8379256225511976D+04
	5	0.7930761925862192D+01	0.7220252827343957D+01	0.1728833046375848D+03	0.1476119398555088D+04
	7	0.5793666406488995D+01	0.1017152618627460D+02	-0.1950170803438382D+02	-0.2094765629128049D+03
	9	0.2259092835829710D+01	0.1327569899026690D+02	-0.8555920522794369D+00	0.8666321495000448D+01
$I_{7,10}$	1	0.1021449035430260D+02	0.4562479433004412D+01	0.1994590873214745D+04	-0.1372458842018080D+05
	3	0.1082098193052004D+02	0.1517953393707028D+01	-0.1093348615695663D+04	0.2429062402918286D+05
	5	0.8932235514655394D+01	0.7637703369347424D+01	-0.1160477727112733D+04	0.4082211495842128D+04
	7	0.6786787372173969D+01	0.1078715258382659D+02	0.2784418667587466D+03	-0.5222613762862302D+03
	9	0.3245504828348013D+01	0.1414179989064441D+02	-0.1920639716510513D+02	0.1452887303079351D+02
$I_{8,10}$	1	0.1122085377938947D+02	0.4792964167579433D+01	0.1408905585646204D+05	-0.3717879118603159D+05
	3	0.1183009373917065D+02	0.1593753005857439D+01	-0.8143311840146222D+04	0.6953735577620818D+05
	5	0.9933383722179631D+01	0.8033106334264012D+01	-0.7314870126205923D+04	0.9590904092538298D+04
	7	0.7781146264463219D+01	0.1136889164904948D+02	0.1435209080513822D+04	-0.8370547420432651D+03
	9	0.4234522494797053D+01	0.1495704378128172D+02	-0.6608297062374048D+02	-0.8738373994399590D+01
$I_{9,10}$	1	0.1222613148416987D+02	0.5012719263664313D+01	0.6127699970619877D+05	-0.9540859890801835D+05
	3	0.1283767707780707D+02	0.1666062584183286D+01	-0.3690204688053926D+05	0.1969904635282219D+06
	5	0.1093430343059445D+02	0.8409672996006852D+01	-0.2891657227016477D+05	0.1816918510004454D+05
	7	0.8776434640084478D+01	0.1192185389830106D+02	0.4655360846405155D+04	-0.1901773025725914D+01
	9	0.5225453367344143D+01	0.1572952904563915D+02	-0.1187414018999123D+03	-0.1413036923216711D+03

Note: All nonreal α_i, K_i occur in complex conjugate pairs. One member of each pair is tabulated, the other is obtained from the relations:

$$\mathrm{Re}(\alpha_i) = \mathrm{Re}(\alpha_{i-1}), \qquad \mathrm{Im}(\alpha_i) = -\mathrm{Im}(\alpha_{i-1})$$

$$\mathrm{Re}(K_i) = \mathrm{Re}(K_{i-1}), \qquad \mathrm{Im}(K_i) = -\mathrm{Im}(K_{i-1})$$

for $i = 2, 4, \ldots, 2[N/2]$, where $[x]$ denotes the greatest integer $\leqslant x$.

Let $\Omega = \{7, 8, 10, 12, 14, 15\}$. Table 3 gives the number

$$\max_{\Delta \in \Omega} (\Delta - n_0) \quad \text{for each } (M, N) \text{ such that } 0 \leqslant M < N \leqslant 10.$$

For each of the values of Δ cited, and for almost all these (M, N) the number $\Delta - n_0$ is either equal to, or one less than, the maximum tabulated.

4. Conclusions. Using the tests of Section 3, it is found that the accuracy of the constants attainable using 15-decimal floating arithmetic is sufficient for practical use whenever $N \leqslant 10$. The constants tabulated are therefore applicable to a wide range of problems. If arithmetic of higher precision is available, the method given may be used readily to extend the tables so as to give constants of higher precision and constants for $N > 10$. Moreover, the tests of Section 3 may again be applied to assess the accuracy of the constants.

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